

# Interface Motion and Pinning in Small World Networks

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We show that the nonequilibrium dynamics of systems with many interacting elements located on a small-world network can be much slower than on regular networks. As an example, we study the phase ordering dynamics of the Ising model on a Watts-Strogatz network, after a quench in the ferromagnetic phase at zero temperature. In one and two dimensions, small-world features produce dynamically frozen configurations, disordered at large length scales, analogous of random field models. This picture differs from the common knowledge (supported by equilibrium results) that ferromagnetic short-cuts connections favor order and uniformity. We briefly discuss some implications of these results regarding the dynamics of social changes.

Small-world networks have received a great deal of attention in the past few years, in particular for their realistic description of the topology of the interactions that take place among populations in various biological, social or economical systems [1]. An important feature of small worlds, that is not shared neither by regular lattices nor random networks, is the interplay that exists between local (or “physical”) interactions, *e.g.* between nearest-neighbors, and non-local ones, involving nodes (or agents) separated by large distances but connected through short-cuts. Among other outstanding topological properties, the effective space dimension of such networks grows linearly with their size [2], even if the fraction of sites with short cuts is very small.

The strong connectivity of small worlds usually enhances dramatically cooperative effects, as predicted by epidemic models of spreading of diseases [4], or of propagation of conventions or rumors in social systems [5]. Naturally, many models of social dynamics have been inspired from the Ising model [6]. The Ising model on a small world exhibits ferromagnetic order at low temperatures even in one dimension [7], while, in higher dimensions, the critical temperature is increased compared with that of the regular lattice [8]. In addition, the fact that the transition is of mean-field nature agrees with the intuitive argument that each site is effectively close to a large number of sites due to the short-cuts of the lattice.

However, because of their inherent random topology, one may ask whether in some situations small-world networks would not rather exhibit features characteristic of disordered systems. In this work, we study as a basic example the nonequilibrium dynamics of the Ising model, as observed after a rapid quench from the high temperature phase to the ferromagnetic phase. We show that the random (all ferromagnetic) connections that enhance ordered states at thermodynamic equilibrium, are responsible in the present case for very slow dynamics and stabilize at large times configurations that, instead of being uniform, are spatially heterogeneous. At zero temperature, systems do not perform long range order dynamically, but remain asymptotically trapped in metastable states characterized by a finite domain size. These fea-

tures are reminiscent of nonequilibrium processes in the random field Ising model (RFIM) [9], in binary mixtures with fixed impurities [10], as well as in a few social models on regular lattices like the voter model [11]. This has to be contrasted with the much more efficient phase ordering kinetics of the Ising model on regular lattices (or Model A in the lexicon of Hohenberg and Halperin [12]), where the mean size of ordered domains grows with time as  $t^{1/2}$  [13]. Our present analysis focuses on the motion of domain walls between “up” and “down” domains, and shows evidence of competing effects between surface tension and pinning (or localizing) effects.

We use a standard model of small-world network [14] consisting of a regular square lattice (or a chain in 1D) composed of  $N$  nodes connected to their nearest neighbors. For each site, we then establish with a probability  $p$  an additional connection, or short-cut, linking the considered site to an other site chosen at random in the lattice. (We do not remove the nearest neighbors connections.) For  $p = 0$ , the lattice is regular, while for  $p = 1$ , the network is strongly disordered. Here, we will consider only the so-called “small world” limit, that corresponds to the case  $p \ll 1$ , where connections are mainly local and only long-ranged for a small fraction of nodes.

On a fixed network, we then assign to each node a spin-like variable  $S_i = \pm 1$ : it represents a social convention, initially chosen at random for each node. At each time step, each node updates its convention in order to reach a better consensus with the nodes it is connected to. In other words, the system follows a zero temperature Glauber dynamics with the Hamiltonian  $H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j$ , where the sum is performed over all possible pairs of nodes.  $J_{ij} = 1$  if sites  $i$  and  $j$  are connected,  $J_{ij} = 0$  otherwise. At each step, a spin is thus chosen at random and flipped. If  $H$  decreases, does not change, or increases, the new configuration is accepted with probability 1, 1/2 and 0, respectively.

In regular networks ( $p = 0$ ), the system evolves toward a minimum of  $H$  (all  $S_i$ 's equal to +1 or -1). Transient configurations are characterized by the presence of growing and competing ordered domains of “up” and “down” spins. The large time dynamics is controlled

by the motion and annihilation of interfaces (or domain walls) that separate these domains. As for many other systems ordering in phases with broken symmetries, its time evolution is self-similar: The two-point correlation function,  $C(r, t) = \langle S_i(t)S_{i+r}(t) \rangle$ , obeys a scaling relation  $C(r, t) = f(r/\xi(t))$ , where  $f$  is a scaling function, while  $\xi$ , the domain size, grows as  $t^{1/2}$  [13].

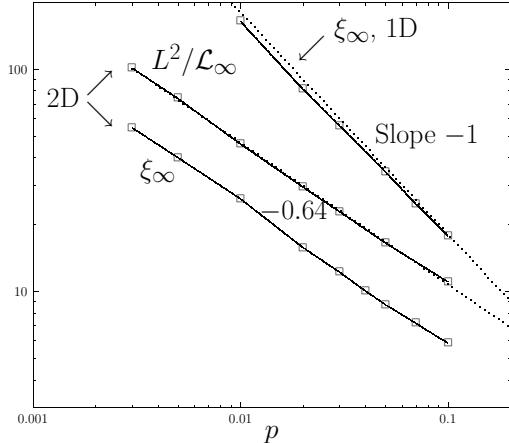


FIG. 1. Asymptotic correlation length  $\xi_0$  in lattice spacing units as a function of the reconnection probability  $p$ , in 1D and 2D.  $L^2/\mathcal{L}_\infty$  is the interface characteristic length in 2D.

On the contrary, in small-world networks ( $p \neq 0$ ,  $p \ll 1$ ) one observes after some time that the typical domain size of ordered spins saturates to a finite value, which decreases when the density of short-cuts ( $p$ ) increases. For a one dimensional chain (of length  $L = 10^5$ ), we plot  $\xi(t = \infty)$  as a function of  $p$  in Figure 1. The correlation length is determined from the half-width of  $C$  averaged over 10 networks and initial conditions. A behavior  $\xi(t = \infty) \propto 1/p$  can be observed.  $1/p$  represents the characteristic size of the one dimensional network, *i.e.* the average distance between two nodes that have long range connections (or “influent” nodes).

Influent sites strongly affect the motion of interfaces. At low  $p$ , most of these nodes are characterized by one additional connection. On Figure 2a, two nodes far apart, A and B, are connected, and  $S_A = -S_B$ . Any interface  $I$  passing through node A leftward can not jump back toward the right, since it is energetically unfavorable. Therefore, at large times, through interface motion, influent nodes will tend to be (irreversibly) connected to nodes that have the same spin (a situation analogous to assortative mixing, as observed in some real life networks [3]).

This argument can be extended to a succession of domains. Figure 2b illustrates a typical large time configuration: In this example, interface  $I_1$  stands between two influent nodes A and B with opposite spins: for the reason mentioned above,  $I_1$  can not jump to the left of A, nor to the right of B. The interface is then localized, *i.e.*

restricted to perform a random walk within the interval  $[A, B]$ . Interface  $I_2$  is therefore unable to annihilate with interface  $I_1$ , that is localized between C and D. Hence, the two disjoint black domains can not merge to form a bigger one, what would happen in the standard Ising model. The domain size, or correlation length  $\xi$ , does not exceed the distance, of order  $1/p$ , that separates influent unlike (antagonist) successive nodes.

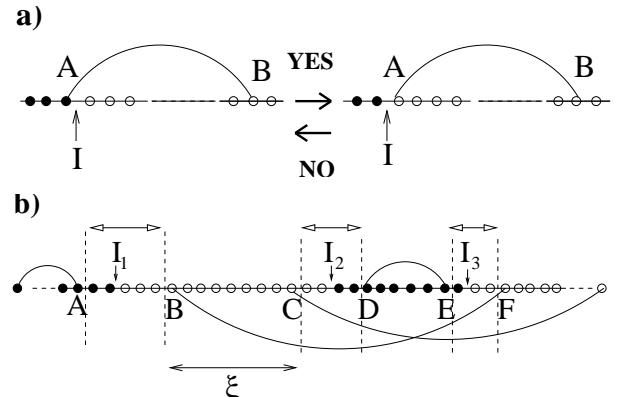


FIG. 2. Up and down domains. Domain walls  $I_n$  become localized in 1D.

We also find that the structure of frozen configurations obeys a scaling relation with  $p$ , *i.e.* that they are statistically independent of  $p$  via proper length rescaling. The asymptotic correlation function  $C(r, t = \infty)$  is plotted in Figure 3 as a function of the reduced variable  $r/\xi_\infty(p)$ , for various values of  $p$ . Data collapse rather well on a single curve. At short times, the kinetics is not affected by the small world structure of the lattice, and that  $\xi(t)$  starts growing as  $t^{1/2}$ . When interfaces become localized, the structure can be roughly seen as the one given by the standard Glauber dynamics of the 1D Ising model stopped at a time  $(1/p)^2$ . Since that problem obeys dynamical scaling, frozen configuration should scale with parameter  $p$ . However, this picture is not quantitatively correct, as the scaling functions in both problems slightly differ.

Finite, low temperature effects are quite subtle in one dimensional small-worlds since they do not destroy the ferromagnetic order observed at  $T = 0$ , unlike in usual Ising chains. One can interpret here the order/disorder transition temperature  $T_c$  as the temperature where ordering via interface jumps over localizing barriers (that enable further domain merging) no longer overcomes disordering happening within domains (subdomain creation). In Figure 2b, interface  $I_2$  (or  $I_1$ ) can jump in interval  $[B, C]$  at a rate  $r_a = \exp(-2/T)$ . Besides, the rate at which any spin among the  $p^{-1}$  spins of interval  $[D, E]$  would flip is  $r_b = p^{-1} \exp(-4/T)$ : it is roughly the rate at which a white domain is created and can start to grow. Qualitatively, the order/disorder transition occurs when  $r_a = r_b$ . This gives:  $T_c \simeq -2/\ln p$ , a expression derived (with a  $\propto$  sign) in ref. [7] using the replica method.

The above relation may be exact as  $p \rightarrow 0$  (as the numerical prefactors in the different rates become irrelevant). Simulation results (not shown) give  $T_c = -2.3/\ln p$  for  $p = 0.03$ .

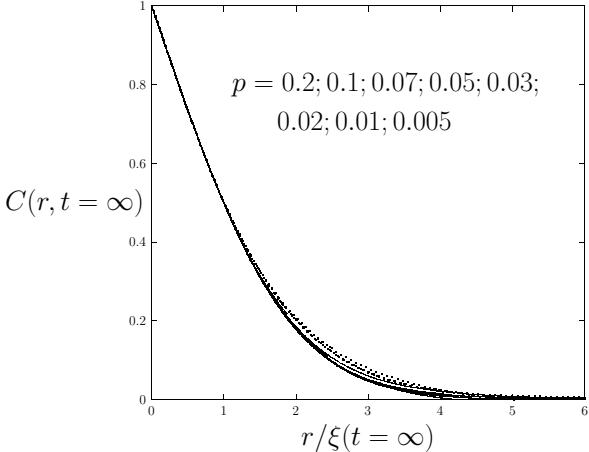


FIG. 3. Two point correlation function in 1D (dotted lines) and 2D (solid lines) as a function of  $r/\xi_\infty$ , for various  $p$ .

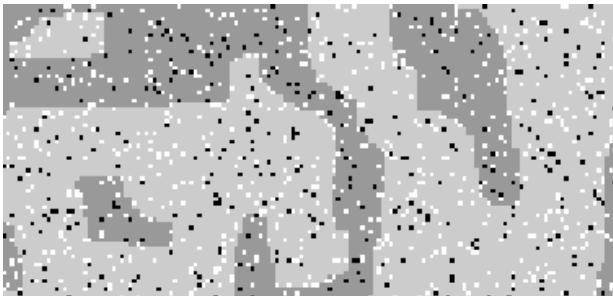


FIG. 4. Frozen domains (in gray) for  $p = 0.05$  (6% of the system total area). The white and black dots represent the “like” and “unlike” influent nodes, respectively.

In two dimensions, one also observes that random initial configurations freeze at large times. Figure 1 displays as a function of  $p$  the asymptotic correlation length (determined from  $C(r, t = \infty)$  averaged over 8 networks with  $1500^2$  spins), as well as the length associated with interface density. The latter is defined as  $L^2/\mathcal{L}$ , where  $L$  is the system linear extend and  $\mathcal{L}$  the total length of all boundaries. Both length scales remain proportional to each other when varying  $p$ , suggesting that frozen configurations can be characterized by one characteristic length scale, referred to as the “domain size”,  $R_\infty(p)$ . Numerical results suggest that  $R_\infty$  varies as an inverse power-law of  $p$ , with a non-trivial exponent close to  $-2/3$  over nearly two decades. Surprisingly,  $R_\infty$  does not scale as  $p^{-1/2}$ , the characteristic length scale of the network [2]. Once again, the spin-spin correlation function at  $t = \infty$  scales rather well as  $C(r) = f(r/\xi_\infty(p))$ , see Figure 3.

Figure 4 shows a typical frozen pattern at  $p = 0.05$ . The positions of the “influent” spins are marked by dots:

in white, those which are connected to an other influent spin of same sign (“like” pairs, of number density  $n_l$ ), in black those connected to a spin of opposite sign (“unlike” pairs, number density  $n_u$ ). Initially,  $n_l \simeq n_u$ , but as coarsening proceeds, “unlike” dots turn more easily to “like” than the contrary, as in one dimension (Fig. 2a). Once again, mixing tends to be assortative ( $n_l > n_u$ ), but with the increase of  $n_l$ , at some point, there are no more possible moves toward better consensus. We find numerically that coarsening stops and interfaces get pinned when  $n_l \simeq 1.86 n_u$ .

The finite domain size can be interpreted as the result of competing effects between surface tension (the driving force for domain growth) and energy barriers created by the multiplication of influent “like” sites. We picture the system as a collection of  $L^2/R^2$  domains of radius  $R$ , and estimate its energy change when domains coarsen from  $R$  to  $R + dR$  ( $dR > 0$ ). The usual contribution from surface tension is  $\delta E_I \propto -2L^2 dR/R^2$ . Meanwhile, the number of influent nodes that flip spin is proportional to  $2pRdR(L^2/R^2)$ . “Like” nodes turn to “unlike” (with an energy cost per spin of 2), and reversely (with an energy decrease of -2). The total energy difference thus reads

$$\delta E \propto \left[ -\frac{2}{R^2} + \frac{n_l - n_u}{n_l + n_u} \frac{4p}{R} \right] L^2 dR. \quad (1)$$

The second term is positive and dominate at large  $R$ . Hence, coarsening is arrested when  $\delta E = 0$ , or  $R_\infty \sim p^{-1}$ . This argument is somehow similar to the (equilibrium) Imry-Ma argument for the RFIM [15]. Yet, an important difference is that here the average magnetic field felt on influent nodes (or “impurities”) is not zero, but has been biased ( $n_l \neq n_u$ ) due to previous spin flips.

The above continuous Imry-Ma-like argument qualitatively explains frozen states, but over-estimates  $R_\infty$  ( $\sim p^{-1}$  instead of  $p^{-2/3}$ ). The exponent  $-2/3$  can be explained as an effect of the square lattice. As shown on Figure 5a, a single influent “like” node located at a domain corner can disappear through the diffusive motion of a step. Figure 5b represents then the simplest distribution of “like” nodes such that the hatched domain can not shrink. It is composed of two right-angle corners  $\{A, A_1, A_2\}$  and  $\{B, B_1, B_2\}$  defining a square  $r \times r$ . If the other white nodes  $\{D_1, \dots, D_n\}$  contained in the square do not form any right-angle corners, then this region encloses the smallest (or “critical”) pinned domain: any bubble of hatched region comprised in the square and that does not contain both corners  $\{A, A_1, A_2\}$  and  $\{B, B_1, B_2\}$  will shrink. Any larger bubble will not. We now calculate the probability  $P_{\text{freeze}}(r)$  that a configuration such as represented in Fig. 5b has a size  $r$ , and then identify  $r^*$  such that  $P_{\text{freeze}}(r^*)$  is maximal with the asymptotic domain size  $R_\infty$  in the disordered medium.

Given the node  $A$  located at the origin, the probability that there is at least one white dot ( $A_1$ ) on the same line within a distance  $r$  is  $P_1(r) = 1 - (1 - p_l)^r$ , with  $p_l/p = n_l/(n_l + n_u)$  the fraction of influent nodes that are

“like” (here, the numerical value of this ratio – close to 0.65 – is unimportant and could be set to 1). Therefore,  $P_{\text{freeze}}(r) = p_l[P_1(r)]^4 P_2(r)$ , with  $P_2$  the probability that the  $D_n$ ’s do not form right-angle corners, *i.e.* that each node  $D_i$  is at least located on a line or a column not occupied by an other  $D_j$  (see the dotted lines in Fig. 5b).  $P_2$  can be approximated as

$$P_2(r) \simeq \sum_{n=0}^{(r-2)^2} (1-p_l)^{(r-2)^2-n} p_l^n C_{(r-2)^2}^n [1-P_1(r)]^n, \quad (2)$$

or  $P_2(r) \simeq (1-p_l P_1(r))^{(r-2)^2}$ . In the sum (2) we have multiplied the probability of having  $n$  white dots inside the square by the probability  $[1-P_1(r)]^n$  that  $n$  independent dots have no neighbors on the same line within  $r$ . For  $n$  small, no or few corners can be formed anyway, so that relation (2) slightly under-estimated  $P_2$ , since a small fraction of empty sites are counted twice ( $[1-P_1(r)]^n \lesssim 1$ ). For  $n$  large, on the contrary, relation (2) over-estimates  $P_2$ , since it is impossible to locate many dots without forming corners (while  $[1-P_1(r)]^n$  is small but  $\neq 0$ ). We suppose that both errors compensate. This factorization enable us to compute the most probable square size  $r^*$  analytically.  $P_1(r)$  increases with  $r$ ,  $P_2(r)$  decreases with  $r$ , and  $P_{\text{freeze}}(r)$  has one single maximum. Assuming  $p \ll 1$ ,  $r \gg 1$ ,  $rp \ll 1$ , we find that  $r^* = p_l^{-2/3} \propto p^{-2/3}$ , in agreement with the numerical results.

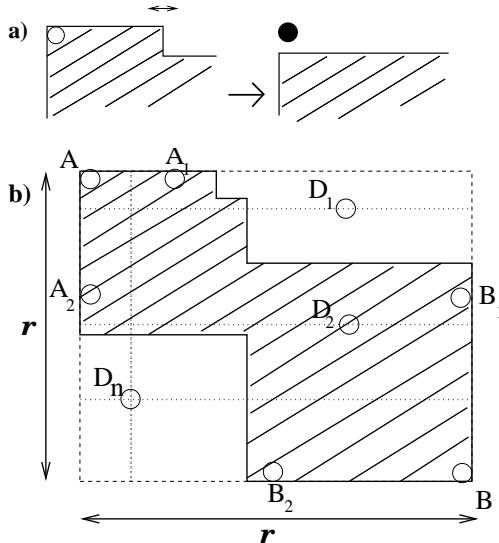


FIG. 5. a) Free and b) pinned domain in presence of “like” influent nodes.

To summarize, we have shown on an example that assortative mixing in small world networks can dynamically generate frozen metastable states : At large times, some influent nodes have simply no immediate interest to evolve. These results suggest that long term dynamics in highly connected social systems can produce spatial

heterogeneities (or segregation), despite that these configurations are not the most desired ones by individual agents. A similar picture, in agreement with some empirical observations, was drawn recently from antiferromagnetic models on scale-free networks [16]. Right after strong political changes (in Eastern European countries in 1989, in Mexico in 2000) the evolution of reforms can be fast, but rapidly social inertia takes over and renders further adjustments difficult or null. Physically speaking, the response of social systems to external forcings (*i.e.* large-scale policies) is susceptible to exhibit some of the interesting features known for disordered systems [9].

While revising the manuscript, we became aware of a similar study on the voter model on small-worlds [17]. We acknowledge fruitful discussions with G. Cocho, J. Viñals and R. Boyer.

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